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On the mathematical foundation of the (1,1,2)-plate model

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Abstract

The plate problem of three-dimensional (3-D), linearized elastostatics is considered in the framework of the hierarchical modelling with the help of the energy projection method. It is shown that the (1,1,2)-bending model is the 'simplest' asymptotically correct model in the hierarchical family, i.e. the distance between the solution of the 3-D problem and the model-solution in the energy norm tends to zero for vanishing plate thickness h while using unmodified, 3-D material laws.

The formulation of the (1,1,2)-bending model and the mechanical significance of its ingredients are discussed. We present error estimates for the deviation of the (1,1,2)-solution from the Kirchhoff solution as well as from the 3-D-solution, by using the (1,1,2)-energy norm and the 3-D-energy norm, respectively. The analysis leads to $\sqrt{h}$ as order of convergence. The results are illustrated by a numerical example.

Keywords: Plate bending; Energy projection method; Polynomial approximation; Asymptotic correctness; 3-D-material laws

1. Introduction

1.1. Remarks on the history of plate modelling

The problem of thin elastic plate bending has been a challenging subject to generations of scientists, both engineers and mathematicians, throughout the past centuries. From the first efforts by Cauchy and Poisson some two hundred years ago, up to higher-order shear deformation theories and the asymptotic analysis for the 3-D formulation, developed in the second half of the twentieth century, there have been innumerable different approaches.

The first usable plate bending theory was presented by Kirchhoff (1850). The theory utilized the assumption that the normal to the mid-plane remains normal during deformation, thus neglecting transverse shear strain effects. In the middle of the present century, Reissner (1944) and Mindlin

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(1951) independently developed plate bending formulations which considered approximately the effect of transverse shear deformations by giving up Kirchhoff's normality constraint. Mindlin's formulation can be interpreted as a (1,1,0)-model, because the displacement assumptions are linear for the two in-plane displacements and constant for the transversal deflection. Both, Kirchhoff's and Mindlin's models are based on the hypothesis $\sigma_{33} = 0$ which implies that a modification of the 3-D material equations is needed to obtain sensible results, i.e. the correct set of differential equations.

The success of plate (and shell) formulations, utilizing a kinematic description of the shell body due to Reissner and Mindlin in the recent past, can be set down to the fact that these are much better suited for the use in a finite element formulation. In the past thirty years there has been a rapid development of corresponding plate and shell finite elements.

Also around the middle of the present century the mathematical analysis of the modelling error and the justification of plate models via asymptotic analysis gained more and more significance. Indeed, Morgenstern (1959) was the first to prove that the Kirchhoff model is the correct asymptotic limit of the 3-D model as the thickness of the plate tends to zero. A systematic investigation of this convergence was done by the group around Ciarlet (see Ciarlet, 1990 for a survey and further references). Thus, it was shown that the Kirchhoff assumptions do not have to be made intuitively but can be derived from 3-D elasticity theory as the limiting case in asymptotic analysis. Moreover, Babuška and Pitkäranta (1990) presented energy error estimates for the deviation of the Kirchhoff solution to the 3-D-solution and the solution of the Mindlin plate model, respectively, using Morgenstern's technique. Many papers since the early sixties dealt with the asymptotic analysis by applying singular perturbation techniques to the 3-D formulation starting with Friedrichs and Dressler (1961) and Gol'denveizer and Kolos (1965). In particular the influence of the boundary layer had to be studied carefully. Shoikhet (1976) derived estimates between the 3-D solution and an approximation via asymptotic analysis even in the case of certain nonlinear constitutive laws by generalizing Morgenstern's approach. More recently a full discussion of the asymptotics of the plate problem for mid-planes with smooth boundaries has been given by Nazarov and Zorin (1989) and Dauge and Gruais (1996, 1998). Moreover, the asymptotics of the Mindlin model was thoroughly analyzed by Arnold and Falk (1996) and by Bathe and Häggblad (1990).

Throughout the history of plate analyses there has been an interest in higher order formulations in order to improve accuracy for the analysis of thick and also laminated plates. Such a formulation was probably presented first by Hildebrand in 1949, see Reissner (1986). He introduced a (1,1,2)-model, i.e. the transversal displacement was assumed to vary quadratically across the thickness. In the sequel there have been attempts to develop even more sophisticated formulations (see Lo et al., 1977 for a 22nd-order theory) up to polynomial expansions of arbitrary order (Naghdi, 1972; among others). However, at that time these plate theories did not gain significance in practical analysis, because they were too elaborate. With the recent developments in computational mechanics, however, these higher order models became more and more important. At the beginning of the nineties the first plate and shell finite elements, using a (1,1,2)-kinematics have been presented (Büchter et al., 1994; Parisch, 1993; Sansour, 1995; among others). Here, the driving force was at first not the improvement of accuracy but the fact that 3-D constitutive laws could be applied without modification. Thus, the door was opened to true 3-D shell analysis, including large strains and the application of any arbitrary material law. On the other hand, the computational efficiency and mechanical clarity of a 2-D formulation has been retained.

This lasting progress in numerical analysis and computer technology also led to a growing interest in creating a hierarchy of models of increasing accuracy and complexity. In the case of homogeneous plates admissible displacement fields that are polynomial with respect to the thickness variable can be used. Hierarchical models obtained by the energy projection method have been studied extensively by Babuška and Li (1991, 1992) and by Schwab (1995, 1996). This approach can be considered as a generalization of the higher-order shear deformation theories. It is well-known that for the lower degree models the constitutive law must be modified in order to have asymptotic correctness. This phenomenon has been studied in some generality by Paumier and Raoult (1997), where it is shown that one requires at least polynomials of degree one for the horizontal displacements and quadratic polynomials for the transversal deflections for the minimum energy models in order to have consistency with the Kirchhoff model in the limit of vanishing thickness. In fact, they proved necessary and sufficient conditions for the asymptotic correctness of minimum energy models, including the (1,1,2)-model as a special case. A more general hierarchy of models based on the Hellinger–Reissner principle is formulated and analyzed by Alessandrini et al. (1998).

1.2. Outline

The present paper focuses on the discussion of the (1,1,2)-model with emphasis on the applicability of unmodified 3-D material laws. We start with the formulation of the plate problem within the framework of 3-D, linearized elastostatics. Dimensional reduction is achieved by energy projection of the 3-D displacement field on a closed subspace of admissible displacements, namely polynomials with respect to the thickness variable (Section 2). Increasing the space of admissible displacements, a hierarchy of nested spaces and models is obtained. This concept is nothing else than a particular Galerkin method, or some p-method within a finite element formulation (see Babuška and Li, 1991). We show that the classical plate models (Kirchhoff, Mindlin) are not members of the hierarchy, although the Mindlin plate model can be understood as the (1,1,0)-model by energy projection if the material law is appropriately modified.

In Section 3 the motivation for a plate model is given which is able to handle unmodified 3-D constitutive laws. As an example of such a model, one special member of the hierarchical family, namely the (1,1,2)-model, is described in detail. This model is already well established in numerical finite element analysis of plates and shells (see e.g. Büchter et al., 1994). Its governing differential equations, namely the Euler equations of the underlying variational principle, are formulated in terms of stress resultants, as it is common in engineering literature. This notation facilitates the discussion of the mechanical significance of similarities and differences to classical plate models, like Kirchhoff's or Mindlin's.

Our main contribution is the refinement of the asymptotic analysis of the (1,1,2)-model for plates and the proof of its asymptotic correctness in terms of rigorous asymptotic estimates given in Section 4. The proof begins with the formulation of the plate problem by means of variational principles formulated in stress resultants and 'generalized strains'. As in the above mentioned proofs of the asymptotic correctness of Kirchhoff's and Mindlin's models (Morgenstern, 1959; Babuška and Pitkäranta, 1990), the theorem of Prager and Synge (1947) is used. This allows us to estimate the error without explicit knowledge of the 3-D solution. The application of the theorem only requires the formulation of admissible stresses and strains, satisfying the equilibrium and geometry equations, respectively, but not necessarily the material law. The calculation of these appropriate statically and geometrically admissible quantities requires the explicit calculation of certain boundary corrector terms, depending on the boundary conditions. In this paper this is done for the case of a hard clamped plate. The last part of the proof consists of finding a correlation between the energy norms of the (1,1,2)-model and the 3-D problem, in order to obtain an error estimate in the 3-D-energy norm.

A numerical example (Section 5) not only confirms the asymptotic behavior of the (1,1,2)-model, but also illustrates its difference to the Mindlin model in the range of thick plates.

2. Hierarchical modelling—the energy projection method

2.1. The 3-D formulation

In what follows, we will make use of index notation together with Einstein's summation convention. Latin indices vary from 1–3, Greek indices from 1–2.

The plate problem is here considered as a boundary value problem of 3-D linearized elastostatics in the domain

$$\Omega := \omega \times \left(-\frac{h}{2}, \frac{h}{2}\right)$$

of thickness h with the lateral boundary surface

$$\Gamma_0 := \gamma \times \left(-\frac{h}{2}, \frac{h}{2}\right), \quad \gamma = \partial\omega.$$

Here $\omega \subset \mathbb{R}^2$, the mid-plane of the plate, denotes a plane, bounded domain with Lipschitz boundary γ. Furthermore, we define the faces of the plate

$$\Gamma_\pm := \omega \times \left\{\pm\frac{h}{2}\right\}.$$

Then the 3-D plate problem consists of finding a displacement field $u(x, y, z) : \Omega \to \mathbb{R}^3$ which satisfies the following governing equations:

(1) Equilibrium conditions:

$$Lu = -\operatorname{div} \sigma[u] = f \quad \text{in } \Omega \tag{1}$$

with the symmetric stress tensor

$$\sigma[u] = \sigma_{ij}[u] e^i \otimes e^j$$

and the volume forces f.

(2) Constitutive equations (Hookes's law):

$$\sigma[u] = A\varepsilon[u] \tag{2}$$

with a fourth order tensor A of elasticities and the linearized strain tensor

$$\varepsilon[u] = \varepsilon_{ij}[u]e^i \otimes e^j, \quad \varepsilon_{ij}[u] = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

(3) Essential and natural boundary conditions at the lateral side. Although a large variety of boundary conditions may be considered, for the sake of simplicity we will restrict ourselves in the following to the case of a plate hard clamped along its lateral side (i.e. Dirichlet conditions on Γ_0)

$$B_0 u = \gamma_0 u = 0 \quad \text{on } \Gamma_0, \tag{3}$$

where γ_0 denotes the trace operator.

(4) Prescribed normal tractions on the faces:

$$\sigma[u] \cdot n = p^{\pm} \quad \text{on } \Gamma_{\pm}, \tag{4}$$

where n denotes the exterior unit normal vector to $\Gamma_{\pm} \subset \partial\Omega$.

It is convenient to write the six components of σ and ε as column vectors in $\mathbb{R}^6$, where the σ_{ij} refer to the canonical basis

$$\sigma(u) = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{13}, \sigma_{23})^{\mathrm{T}}, \quad \varepsilon(u) = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{13}, \varepsilon_{23})^{\mathrm{T}}.$$

Then Hooke's law (2) for homogeneous and isotropic materials can be written in simplified form

$$\sigma(u) = A\varepsilon(u) := \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\mu \end{bmatrix} \varepsilon(u), \tag{5}$$

where $\lambda = [Ev/((1-2v)(1+v))]$ and $\mu = [E/(2(1+v))]$ are the Lamé constants (v denotes Poisson's ratio and E Young's modulus). In this notation A is a symmetric, positive definite 6×6 matrix.

The variational solution of the plate problem is, according to the principle of minimum potential energy of elastostatics, the displacement field $u : \Omega \to \mathbb{R}^3$ minimizing the potential energy

$$\Pi(u) = \tfrac{1}{2}\mathscr{B}(u, u) - \mathscr{F}(u) \tag{6}$$

within the set

$$\mathscr{H}(\Omega) = \{u \in [H^1(\Omega)]^3 \mid u = 0 \text{ on } \Gamma_0\} \tag{7}$$

of admissible displacement fields. Here the symmetric bilinear form $\mathscr{B}(\cdot,\cdot)$ is given by

$$\mathscr{B}(u,v) = \int_\Omega \varepsilon[v]:\sigma[u]\,\mathrm{d}V \quad \text{for } \sigma,\varepsilon \in [L^2(\Omega)]^3. \tag{8}$$

For given surface tractions $p^\pm \in [L^2(\omega)]^3$ on the faces $\Gamma_\pm$ and volume forces $f \in [L^2(\Omega)]^3$, we define the linear and continuous load functional $\mathscr{F}(\cdot)$ by

$$\mathscr{F}(u) = \int_\Omega u(x,y,z)^\mathrm{T} f(x,y,z)\,\mathrm{d}V + \int_\omega \left\{ p^+(x,y)^\mathrm{T} u\left(x,y,\frac{h}{2}\right) + p^-(x,y)^\mathrm{T} u\left(x,y,-\frac{h}{2}\right)\right\}\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

Clearly, $\mathscr{H}(\Omega)$ is a closed, linear subspace of $[H^1(\Omega)]^3$ and, thus, a Hilbert space equipped with the energy scalar product $\mathscr{B}(\cdot,\cdot)$ and the corresponding energy norm $\|u\|_{e(3\text{-D})} := (\mathscr{B}(u,u))^{1/2}$ (see Theorem 2.1 in Schwab, 1996). Since here the variational and weak formulations are equivalent, each minimizer $u \in \mathscr{H}(\Omega)$ of (6) satisfies the equations in weak formulation

$$\mathscr{B}(u,v) = \mathscr{F}(v) \quad \forall v \in \mathscr{H}(\Omega). \tag{10}$$

Existence and uniqueness of a weak solution of (10) then follow from the Riesz representation theorem in $\mathscr{H}(\Omega)$. We refer to Schwab (1995, 1996) for details concerning existence and uniqueness of variational or weak solutions for different boundary conditions (3) on the lateral side.

The variational solution $u \in \mathscr{H}(\Omega)$ of the plate problem can be decomposed into a membrane part $u^\mathrm{I}(x,y,z)$ and a bending part $u^\mathrm{II}(x,y,z)$ in the following manner:

$$u^\mathrm{I}_\alpha(x,y,z) = u^\mathrm{I}_\alpha(x,y,-z), \quad u^\mathrm{I}_3(x,y,z) = -u^\mathrm{I}_3(x,y,-z)$$

and

$$u^\mathrm{II}_\alpha(x,y,z) = -u^\mathrm{II}_\alpha(x,y,-z), \quad u^\mathrm{II}_3(x,y,z) = u^\mathrm{II}_3(x,y,-z).$$

The corresponding spaces of admissible displacement fields will be denoted by $\mathscr{H}^\mathrm{I}(\Omega)$ and $\mathscr{H}^\mathrm{II}(\Omega)$, respectively. These subspaces of $\mathscr{H}(\Omega)$ are closed and orthogonal with respect to the scalar product $\mathscr{B}(\cdot,\cdot)$ on $\mathscr{H}(\Omega)$, i.e.

$$\mathscr{H}(\Omega) = \mathscr{H}^\mathrm{I}(\Omega) \oplus \mathscr{H}^\mathrm{II}(\Omega) \quad \text{or} \quad \mathscr{B}(u,v) = 0 \quad \forall u \in \mathscr{H}^\mathrm{I}(\Omega), \quad v \in \mathscr{H}^\mathrm{II}(\Omega).$$

Thus, $u^\mathrm{I}(x,y,z)$ and $u^\mathrm{II}(x,y,z)$ can be obtained independently of each other provided the load functional $\mathscr{F}(u)$ can be split correspondingly

$$\mathscr{F}(u) = \mathscr{F}^\mathrm{I}(u) + \mathscr{F}^\mathrm{II}(u).$$

The corresponding membrane and bending loads are given by

$$f^\mathrm{I}_\alpha(x,y,z) = \tfrac{1}{2}(f_\alpha(x,y,z) + f_\alpha(x,y,-z)), \quad f^\mathrm{I}_3(x,y,z) = \tfrac{1}{2}(f_3(x,y,z) - f_3(x,y,-z)),$$

$$p^\mathrm{I}_\alpha(x,y) = \tfrac{1}{2}(p^+_\alpha(x,y) + p^-_\alpha(x,y)), \quad p^\mathrm{I}_3(x,y) = \tfrac{1}{2}(p^+_3(x,y) - p^-_3(x,y))$$

and

$$f_\alpha^{\mathrm{II}}(x,y,z) = \tfrac{1}{2}(f_\alpha(x,y,z) - f_\alpha(x,y,-z)), \quad f_3^{\mathrm{II}}(x,y,z) = \tfrac{1}{2}(f_3(x,y,z) + f_3(x,y,-z)),$$

$$p_\alpha^{\mathrm{II}}(x,y) = \tfrac{1}{2}(p_\alpha^+(x,y) - p_\alpha^-(x,y)), \quad p_3^{\mathrm{II}}(x,y) = \tfrac{1}{2}(p_3^+(x,y) + p_3^-(x,y)).$$

2.2. *Hierarchical plate models*

A method often used for deriving a hierarchy of plate models is the projection method, whereby the reduced models are obtained by projecting the 3-D displacement field on a closed subspace of admissible displacements, namely displacements that are of special form with respect to the thickness variable. Therefore, let $\mathbf{n} \in \mathbb{N}_0^3$ be a three-vector of non-negative integers. We approximate each component $u_i(x,y,z)$ of the displacement field $u(x,y,z)$ by an asymptotic series of the form

$$u_i^{\mathbf{n}}(x,y,z) = \sum_{k=0}^{n_i} X_{ik}^{\mathbf{n}}(x,y)\psi_{ik}\left(\frac{2z}{h}\right),$$

where in general $\psi_i = \{\psi_{ik}(z)\}_{0 \leqslant k \leqslant n_i}$, denote vectors of $n_i + 1$ linearly independent basis functions in $H^1(-1,1)$. For the energy projection method we choose a Legendre series in z of degrees less than or equal to n_i. With increasing $\mathbf{n}$, this creates a hierarchy of models. Let us denote by u_h and $u_h^{\mathbf{n}}$ the solution of the 3-D problem and the $\mathbf{n}$-model of the plate with thickness h, respectively. Then we require the hierarchy of models to satisfy the following properties.

(A) The exact solutions of the hierarchical models $u_h^{\mathbf{n}}$ converge to the exact solution of the problem of elasticity u_h for fixed h:

$$\frac{\|u_h - u_h^{\mathbf{n}}\|}{\|u_h\|} \to 0 \quad \text{for } \mathbf{n} \to \infty, \tag{11}$$

where $\mathbf{n} \to \infty$ means n_1, n_2, $n_3 \to \infty$. If this property is fulfilled we say that the hierarchy is consistent.

(B) The exact solution of every hierarchical model $u_h^{\mathbf{n}}$ converges for every fixed $\mathbf{n}$ to the same limit as the exact solution of the problem of elasticity u_h with plate thickness h when h approaches zero:

$$\frac{\|u_h - u_h^{\mathbf{n}}\|}{\|u_h\|} \to 0 \quad \text{for } h \to 0. \tag{12}$$

In this case we say that the hierarchical model is asymptotically correct. In Paumier and Raoult (1997) for property (B) the word ‘consistent’ is used, but in the authors’ opinion this expression is more suitable for property (A).

Preferably but not necessarily we would like to have

$$\|u_h - u_h^{\mathbf{n}}\| \leqslant \|u_h - u_h^{\mathbf{m}}\| \quad \text{for } \mathbf{n} \succeq \mathbf{m},$$

where we write $\mathbf{n} \succeq \mathbf{m}$ iff $n_i \geqslant m_i$, $i = 1,2,3$, for $\mathbf{n}, \mathbf{m} \in \mathbb{N}_0^3$. In (11) and (12) mostly—but not exclusively—the energy norm is considered.

Let us describe now the energy projection method to get a hierarchy of plate models. Here, hierarchical plate models are obtained by semidiscretization of the plate problem (10) in transverse direction and energy projection. The function $u^{\mathbf{n}}(x, y, z)$, the solution of the dimensionally reduced model of order $\mathbf{n}$, is any minimizer of the total energy $\Pi(u)$ in (6) within the subspace $\mathscr{H}(\mathbf{n}) \subset \mathscr{H}(\Omega)$ of admissible displacement fields of the form

$$u_i^{\mathbf{n}}(x, y, z) = \sum_{k=0}^{n_i} X_{ik}^{\mathbf{n}}(x, y)\psi_{ik}\left(\frac{2z}{h}\right) = X_i^{\mathbf{n}}(x, y)^{\mathrm{T}}\psi_i\left(\frac{2z}{h}\right), \quad i = 1, 2, 3, \tag{13}$$

which is in fact the Galerkin approximation of the 3-D displacement field within the space $\mathscr{H}(\mathbf{n})$. Here the coefficient functions $X_{ik}^{\mathbf{n}}(x, y) \in H^1(\omega)$ may be considered as to be generalized rotations and deflections. In the case of homogeneous materials with the constitutive law (5) we select Legendre polynomials of degree k as director functions $\psi_{ik}(2z/h) = L_k(2z/h)$. The space $\mathscr{H}(\mathbf{n}) \subset \mathscr{H}(\Omega)$ is a closed, linear subspace and, thus, a Hilbert space (see also Proposition 3.1 in Schwab, 1996). This implies again for every $\mathbf{n}$ existence and uniqueness of a dimensionally reduced solution $u(\mathbf{n}) \in \mathscr{H}(\mathbf{n})$. Hence we have

Lemma 2.1.:

For every $\mathbf{n} \in \mathbb{N}_0^3$ there exists a uniquely determined, dimensionally reduced solution $u(\mathbf{n}) \in \mathscr{H}(\mathbf{n})$.

In the following proposition we recall some basic properties of the hierarchical plate models from Proposition 3.2 in Schwab (1996).

Proposition 2.1.:

(1) Optimality of the $\mathbf{n}$-model. There holds the estimate

$$\|u - u(\mathbf{n})\|_{e(\text{3-D})} \leqslant \|u - v\|_{e(\text{3-D})} \quad \forall v \in \mathscr{H}(\mathbf{n}). \tag{14}$$

(2) Let $\mathbf{n} \succeq \mathbf{m}$. Then there holds

$$\|u - u(\mathbf{n})\|_{e(\text{3-D})} \leqslant \|u - u(\mathbf{m})\|_{e(\text{3-D})}, \tag{15}$$

i.e. an increase of the model order never increases the modelling error, which is the preferable property mentioned above.

(3) Convergence of the hierarchy of $\mathbf{n}$-models towards the 3-D problem at any fixed, positive thickness h:

$$\lim_{\mathbf{n} \to \infty} \|u - u(\mathbf{n})\|_{e(\text{3-D})} = 0. \tag{16}$$

This is the consistency condition (A).

The hierarchical plate models are obtained by energy projection onto $\mathscr{H}(\mathbf{n})$ and can consequently be split into a membrane part $u^{\mathrm{I}}(\mathbf{n})$ and a bending part $u^{\mathrm{II}}(\mathbf{n})$ as the 3-D solution. The membrane and bending parts are obtained independently of each other by the solutions of

$$u^j(\mathbf{n}) \in \mathscr{H}^j(\mathbf{n}) \quad \mathscr{B}(u^j(\mathbf{n}), v) = \mathscr{F}^j(v) \quad \forall v \in \mathscr{H}^j(\mathbf{n}) \tag{17}$$

with $\mathscr{H}^j(\mathbf{n}) = \mathscr{H}(\mathbf{n}) \bigcap \mathscr{H}^j(\Omega)$, $j \in \{\mathrm{I}, \mathrm{II}\}$. In general, because of the symmetry properties of the Legendre polynomials, we have the following model orders in dependence on the maximal polynomial degree q in the z-direction

$$\mathbf{n} = (2[q/2], 2[q/2], 2[(q-1)/2]+1) \quad \text{for } j = \mathrm{I},$$

Table 1
Model orders $\mathbf{n}$ for membrane and bending models in dependence on the maximal polynomial degree q

j	$q=1$	$q=2$	$q=3$	$q=4$	$q=5$	$q=6$
I	(0,0,1)	(2,2,1)	(2,2,3)	(4,4,3)	(4,4,5)	(6,6,5)
II	(1,1,0)	(1,1,2)	(3,3,2)	(3,3,4)	(5,5,4)	(5,5,6)

$$\mathbf{n} = (2[(q-1)/2]+1, 2[(q-1)/2]+1, 2[q/2]) \quad \text{for } j = \text{II},$$

where $[x]$ denotes the Gaussian brackets, i.e. the largest integer $\leqslant x$.

Although Mindlin's and Kirchhoff's model can be derived by the energy projection method, the embedding of these classical models into our hierarchy (13) fails. Following Mindlin (1951), we consider the bending of a plate (i.e. $j = \text{II}$) subject to normal loading $p(x, y)$ (see Fig. 1), i.e.

$$f \equiv 0, \quad p^{\text{I}} \equiv 0, \quad p_{\alpha}^{\text{II}} \equiv 0, \quad p_3^{\text{II}}(x, y) = p(x, y). \tag{18}$$

Motivated by the ansatz for the displacement field in the Mindlin model one would suspect that this model coincides with the hierarchical bending model of order $\mathbf{n} = (1{,}1{,}0)$. This, however, is

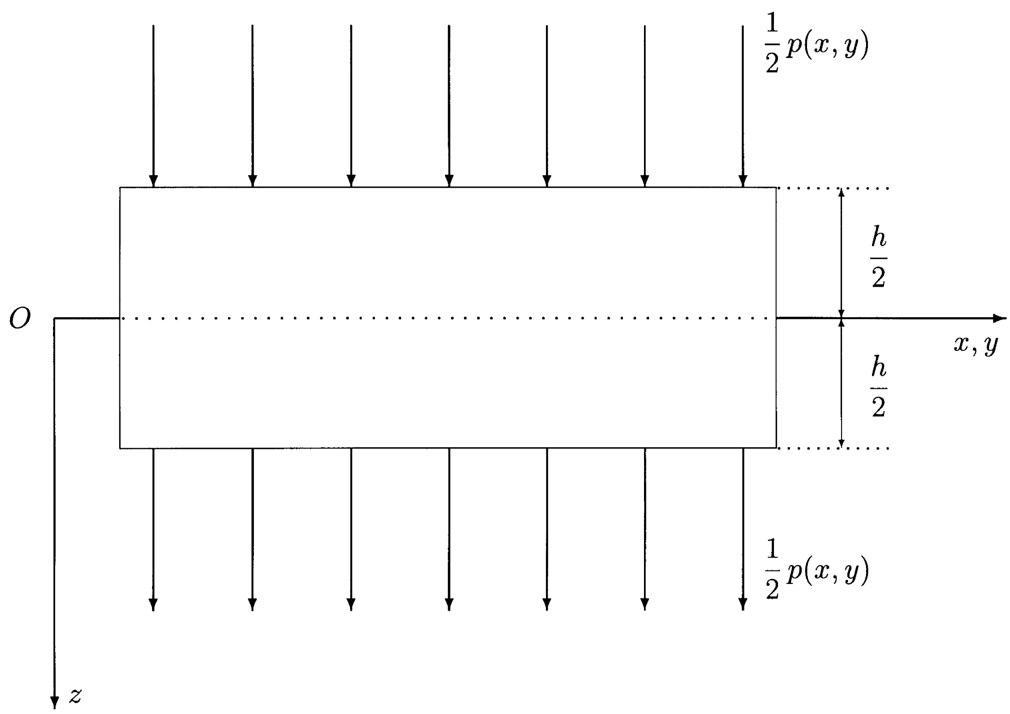

Fig. 1. Thin plate under bending.

not the case. On the other hand, the (1,1,0)-model of the hierarchy is not asymptotically correct, i.e. it does not satisfy condition (12). Paumier and Raoult (1997) investigated this fact in a more general setting.

2.3. Hierarchical models with modified constitutive laws

In order to be able to embed the classical Kirchhoff and Mindlin models into a hierarchical family, the hierarchical models corresponding to (13) are extended in the sense that one allows for modifications of the material law. These modifications should be carried out in a way that property (B) is satisfied. Hence, if we want to guarantee condition (12) for the (1,1,0)-model of an extended hierarchy as well, it has to coincide e.g. with the Mindlin model. If we choose the representation

$$\mathscr{H}^{\mathrm{II}}(1,1,0) = \{u \in \mathscr{H}(\Omega) \,|\, u_1 = z\theta_x, \quad u_2 = z\theta_y, \quad u_3 = w \quad \text{with } \theta_x, \theta_y, w \in H^1(\omega)\},$$

and replace the matrix A in Hooke's law (5) by

$$\tilde{A} = \begin{bmatrix} \Lambda+2\mu & \Lambda & 0 & 0 & 0 & 0 \\ \Lambda & \Lambda+2\mu & 0 & 0 & 0 & 0 \\ \Lambda & \Lambda & 2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\kappa\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\kappa\mu \end{bmatrix} \tag{19}$$

with the modified Lamé constant $\Lambda = (2\mu\lambda)/(2\mu+\lambda)$ and a single shear correction factor $\kappa > 0$, then we obtain the Mindlin-system as the Euler–Lagrange equations of the energy projection within the space $\mathscr{H}^{\mathrm{II}}(1,1,0)$. Note that the requirement of asymptotic correctness for the (1,1,0)-ansatz implies a specific class of material matrices which contains $\tilde{A}$ in (19).

Introducing the space of admissible Kirchhoff displacements

$$\mathscr{H}_K = \{u : \Omega \to \mathbb{R}^3 \,|\, u_1 = -z\,\partial_x w, u_2 = -z\,\partial_y w, u_3 = w \text{ with } w \in H^2(\omega)\},$$

we get the Kirchhoff model as the minimization problem for the potential energy $\Pi(u)$ with the modified matrix $\tilde{A}$ in Hooke's law within the set $\mathscr{H}_K$. The Euler–Lagrange equation for $w(x, y)$ is the classical plate equation

$$D\Delta^2 w = p \quad \text{in } \omega, \tag{20}$$

where the flexural rigidity of the plate D is given by $D = [h^3(\Lambda+2\mu)]/12$. But, if we insert the 3-D-displacements corresponding to the Kirchhoff or the Mindlin solution into the 3-D-total energy Π, this expression does not coincide with the total Kirchhoff or Mindlin energy, respectively. We emphasize that the Kirchhoff model cannot be obtained as a member of the hierarchy without either imposing the Kirchhoff constraint in the trial space, or taking into account additional limit considerations ($h \to 0$).

3. Plate models for 3-D constitutive laws

3.1. Motivation

It is well known, and also clear from the previous remarks that the Mindlin plate model requires a modification of the material law to ensure convergence to the 3-D solution with decreasing thickness. Apart from the simple introduction of a shear correction factor, this modification consists of a condensation of the material law. The assumption

$$\sigma_{33} = 0$$

leads to

$$\varepsilon_{33} = \frac{-\lambda}{\lambda + 2\mu}(\varepsilon_{11} + \varepsilon_{22})$$

and therefore viz (19)

$$\sigma_{11} = \left(\frac{2\mu\lambda}{2\mu + \lambda} + 2\mu\right)\varepsilon_{11} + \left(\frac{2\mu\lambda}{2\mu + \lambda}\right)\varepsilon_{22}$$

$$\sigma_{22} = \left(\frac{2\mu\lambda}{2\mu + \lambda}\right)\varepsilon_{11} + \left(\frac{2\mu\lambda}{2\mu + \lambda} + 2\mu\right)\varepsilon_{22}.$$

Note that now the resulting material law is described by a 5×5 matrix, whose components are not identical to those given in (5), although the resulting differential equations are equivalent leading to an asymptotically correct model. However, these simple algebraic manipulations are only valid for a linearly elastic material description. For more sophisticated constitutive models, as e.g. nonlinear material laws, the analytical condensation of σ_{33} may be difficult to perform or even impossible. The numerical realization of this procedure may on the other hand be computationally expensive, if possible at all. If such material models are required for a sensible finite element analysis of certain structural problems, then 3-D, so-called bricks with a linear displacement field across the thickness can be used. On the other hand, these elements have the disadvantage of being computationally much more expensive than usual plate or shell elements, and behave badly for thin-walled structures due to 'Poisson-locking'.

Therefore, one needs plate (or shell) models which can be used together with unmodified 3-D constitutive laws. The simplest model in the above hierarchy of plate models with this property is the (1,1,2)-model. The first reference, known to the authors, where such a model is introduced is due to Hildebrand, Reissner and Thomas in 1949, a remark can be found in Reissner (1986). Certainly, at that time the motivation for developing such models was different from ours. However, 3-D plate and shell models did not really gain significance until the development of the finite element method facilitated structural analyses with complicated material laws.

In the meantime, the (1,1,2)-model is an established model in large strain shell analysis with finite elements. Three-dimensional shell models have been successfully applied to both, linear and nonlinear analyses of plates and shells (see e.g. Büchter et al., 1994; Bischoff and Ramm, 1997; Parisch, 1993; Sansour, 1995). For plates, consisting of linearly elastic, isotropic, homogeneous materials, a full asymptotic analysis of the (1,1,2)-model (as in Arnold and Falk, 1996 for Mindlin's model) including the computation of the boundary layer terms is given in Alessandrini (1991). A finite element method for the (1,1,2)-plate model including an error analysis is described in Alessandrini (1991) and Alessandrini and Falk (1992). Unfortunately, such a mathematical background is not yet available for shells.

3.2. *The (1,1,2)*-plate model

In this section we give the derivation of the (1,1,2)-model for plates as a member of the hierarchy described in Section 2.2. Since we are only interested in the case of bending and due to technical reasons we restrict ourselves to the special load case described in (18). The mathematical formulation of this model rests on the displacement assumption according to the definition, given in (13), i.e.

$$\mathscr{H}^{\mathrm{II}}(1,1,2) = \{u \in \mathscr{H}(\Omega) \,|\, u_1 = z\theta_x, \quad u_2 = z\theta_y, \quad u_3 = w_0 + z^2 w_2\} \tag{21}$$

with θ_x, θ_y, w_0, $w_2 \in H^1(\omega)$. The linearized strain tensor is obtained from the partial derivatives of the displacements

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} = z\frac{\partial \theta_x}{\partial x}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial y} = z\frac{\partial \theta_y}{\partial y}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial z} = 2zw_2,$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) = \frac{1}{2}\left(z\frac{\partial \theta_x}{\partial y} + z\frac{\partial \theta_y}{\partial x}\right),$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x}\right) = \frac{1}{2}\left(\theta_x + \frac{\partial w_0}{\partial x} + z^2\frac{\partial w_2}{\partial x}\right),$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial z} + \frac{\partial u_3}{\partial y}\right) = \frac{1}{2}\left(\theta_y + \frac{\partial w_0}{\partial y} + z^2\frac{\partial w_2}{\partial y}\right).$$

Basically, the strain tensor is a function of x, y and z. To obtain a 2-D description (x, y) of the strain state, one may introduce equivalent 'generalized strains'. These are the (physical) curvatures, corresponding to bending and twisting, respectively,

$$\kappa_{xx} := \frac{\partial \theta_x}{\partial x}, \quad \kappa_{yy} := \frac{\partial \theta_y}{\partial y}, \quad \kappa_{xy} := \frac{\partial \theta_x}{\partial y} + \frac{\partial \theta_y}{\partial x}, \tag{22}$$

the transverse shear strains

$$\gamma_x := \theta_x + \frac{\partial w_0}{\partial x}, \quad \gamma_y := \theta_y + \frac{\partial w_0}{\partial y}, \quad \delta_x := \frac{\partial w_2}{\partial x}, \quad \delta_y := \frac{\partial w_2}{\partial y} \tag{23}$$

and a normal strain component in z-direction

$$\beta := 2w_2 \tag{24}$$

representing the thickness change during deformation. These are the kinematic relations.

As in a classical plate formulation, stress resultants can be defined by integrating the components of the 3-D stress tensor across the thickness. Thus, the energetically conjugated values to the 'generalized strains' are given by the bending and twisting moments

$$m_{xx} := \int_{-h/2}^{h/2} z\sigma_{11}\,\mathrm{d}z, \quad m_{yy} := \int_{-h/2}^{h/2} z\sigma_{22}\,\mathrm{d}z, \quad m_{xy} := \int_{-h/2}^{h/2} z\sigma_{12}\,\mathrm{d}z,$$

the transverse shear forces

$$q_x := \int_{-h/2}^{h/2} \sigma_{13}\,\mathrm{d}z, \quad q_y := \int_{-h/2}^{h/2} \sigma_{23}\,\mathrm{d}z, \quad r_x := \int_{-h/2}^{h/2} z^2\sigma_{13}\,\mathrm{d}z, \quad r_y := \int_{-h/2}^{h/2} z^2\sigma_{23}\,\mathrm{d}z$$

and the normal stress in thickness direction

$$v := \int_{-h/2}^{h/2} z\sigma_{33}\,\mathrm{d}z.$$

The extension with respect to Mindlin's (1,1,0)-model predominantly influences stresses in thickness direction. In particular, the polynomial order of the shear stress distribution is increased, and a new transverse normal stress component shows up. The moments are identical to those of the Mindlin plate model. It is remarkable that for homogeneous materials the transverse shear stresses σ_{13} and σ_{23} vary quadratically across the thickness, instead of constant, as in Mindlin's model. Thus, a more realistic modelling of the transverse shear stress distribution is achieved. In fact, this is the apparent explanation why with this model no shear correction factor is needed.

It should also be noted that v is not a stress resultant in the conventional sense, because the corresponding cross section is not parallel to the direction of integration! Thus, v must not be described as a force, or couple, like for example q_x or m_{xx}, respectively, but merely as an 'integrated stress component'. The higher order shear terms r_x and r_y, on the other hand, are self-equilibrated 'true' stress resultants, although they do not show up in classical plate models.

With these definitions at hand, the equations of static equilibrium are given by

$$\begin{aligned}
\frac{\partial m_{xx}}{\partial x} + \frac{\partial m_{xy}}{\partial y} - q_x &= 0 \\
\frac{\partial m_{xy}}{\partial x} + \frac{\partial m_{yy}}{\partial y} - q_y &= 0 \\
\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p &= 0 \\
\frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} - 2v + \frac{h^2}{4} p &= 0.
\end{aligned} \tag{25}$$

The first three equations correspond exactly to the Mindlin plate model. Only the fourth equation, representing the equilibrium of forces in direction of thickness contains the new stress resultants and is decoupled from the others. Inserting the relations given by the material law, compare (26), and the kinematic eqns (22)–(24) into the static equilibrium (25) leads to a system of elliptic differential equations for the 'generalized displacements' θ_x, θ_y, w_0 and w_2 of the (1,1,2)-plate model, which is nothing but the system of Euler–Lagrange equations for the energy projection onto the space $\mathscr{H}^{\mathrm{II}}(1,1,2)$.

4. Asymptotic analysis of the (1,1,2)-model

In order to prove that the (1,1,2)-model is asymptotically correct, the same techniques are applied as in Morgenstern (1959) and Babuška and Pitkäranta (1990). First, let us summarize some basic characteristics of variational formalisms and energy principles associated with the (1,1,2)-plate-bending model, which will be needed in the proof. In analogy with the 3-D formulation these will be called the principle of minimum potential energy (energy principle) and the principle of minimum stress energy (complementary energy principle). We will use this terminology for the formulation of the variational principles of the Kirchhoff model as well. The application of the two energy principles or theorem of Prager and Synge (1947) to justify plate theories was initiated in the pioneering work of Morgenstern (1959), where it was used to prove convergence of the 3-D solution towards the Kirchhoff solution when the thickness of the plate tends to zero. We will formulate the theorem of Prager and Synge (1947) for the (1,1,2)-model, which will be used to estimate the distance between the Kirchhoff and the (1,1,2)-solution in the energy norm.

For the sake of simplicity, we only consider the case of the bending of a hard clamped plate under normal surface loading $p(x,y)$ compare (18) and Fig. 1. Other cases can be handled similarly, although technical difficulties may show up. In this simple case the geometrically admissible quantities for the (1,1,2)-model are given by the space $V_{(1,1,2)} = [H_0^1(\omega)]^4$. In the following we will give a formulation, in which we already have divided by the factor h^3. Therefore let $p = h^3 g$. Thus, we have guaranteed that the solution of the (1,1,2)-model converges for $h \to 0$ towards an expression, which neither vanishes nor is infinite. The scaled stress–strain-relation corresponding to this formulation then reads as follows

$$(m, v, q, r)^{\mathrm{T}} = T(\kappa, \beta, \gamma, \delta)^{\mathrm{T}}, \tag{26}$$

where the tensor $T\colon [L^2(\omega)]^8 \to [L^2(\omega)]^8$ is given by the matrix

$$T = \frac{1}{12}\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (12\mu)h^2 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & (3h^2\mu)/20 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (12\mu)/h^2 & \mu \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & (3h^2\mu)/20 \end{bmatrix},$$

which is invertible for $\mu > 0$, $(3\lambda+2\mu) > 0 (E > 0, -1 < \nu < 0.5)$, and T and T^{-1} are real and self-adjoint. For brevity we have used in (26) for the eight-vectors the notation

$$(m, v, q, r) = (m_{xx}, m_{yy}, v, m_{xy}, q_x, r_x, q_y, r_y)$$

$$(\kappa, \beta, \gamma, \delta) = (\kappa_{xx}, \kappa_{yy}, \beta, \kappa_{xy}, \gamma_x, \delta_x, \gamma_y, \delta_y).$$

This notation will be extensively used in the following. Let us further introduce the two symmetric bilinear forms:

$$\mathscr{B}_{(1,1,2)}(w_0, w_2, \theta_x, \theta_y\,; \tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y) = ([\kappa, \beta, \gamma, \delta]^{\mathrm{T}}, T[\tilde{\kappa}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}]^{\mathrm{T}})_{L^2(\omega)},$$

where $[\kappa, \beta, \gamma, \delta]^{\mathrm{T}}$, $[\tilde{\kappa}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}]^{\mathrm{T}}$, are to be expressed by (22)–(24), w_0, w_2, θ_x, θ_y, $\tilde{w}_0$, $\tilde{w}_2$, $\tilde{\theta}_x$, $\tilde{\theta}_y \in H_0^1(\omega)$; and

$$\mathscr{A}_{(1,1,2)}([m, v, q, r]\,; [\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}]) = ([m, v, q, r]^{\mathrm{T}}, T^{-1}[\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}]^{\mathrm{T}})_{L^2(\omega)}$$

with m, $\tilde{m} \in [L^2(\omega)]^3$; q, r, $\tilde{q}$, $\tilde{r} \in [L^2(\omega)]^2$; v, $\tilde{v} \in L^2(\omega)$. Here $(\cdot,\cdot)_{L^2(\omega)}$ is the $L^2(\omega)$-scalar product. Finally, we define the load functional

$$\mathscr{F}_{(1,1,2)}(w_0, w_2, \theta_x, \theta_y) = \int_\omega g\left(w_0 + \frac{h^2}{4} w_2\right) \mathrm{d}x\, \mathrm{d}y$$

for $g \in L^2(\omega)$. Then the (1,1,2)-formulation of the principle of minimum potential energy states:

Find the quadruple $(w_0, w_2, \theta_x, \theta_y)^{\mathrm{T}} \in V_{(1,1,2)}$ minimizing the functional

$$\Pi_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y) = \tfrac{1}{2}\mathscr{B}_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y\,; \tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y) - \mathscr{F}_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y) \quad (27)$$

within the space $V_{(1,1,2)}$ for given $g \in L^2(\omega)$. Existence and uniqueness of $(w_0, w_2, \theta_x, \theta_y)^{\mathrm{T}} \in V_{(1,1,2)}$ follow immediately from the derivation of the (1,1,2)-model as hierarchical model, see Lemma 2.1. The complementary energy principle for the (1,1,2)-model reads:

Find the 8-tuple $(m, v, q, r)^{\mathrm{T}} \in [L^2(\omega)]^8$ minimizing the functional

$$\Pi^C_{(1,1,2)}([\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}]) = \tfrac{1}{2}\mathscr{A}_{(1,1,2)}([\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}]\,; [\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}]) \quad (28)$$

within the space $[L^2(\omega)]^8$ under the constraints (25) with g instead of p. Stress resultants $(m, v, q, r)^{\mathrm{T}} \in [L^2(\omega)]^8$ satisfying the equilibrium conditions (25) with g instead of p in the domain ω are called statically admissible. We denote by $\Sigma_{(1,1,2)} \subset [L^2(\omega)]^8$ the space of all statically admissible stress resultants. For the solutions $(w_0, w_2, \theta_x, \theta_y)^{\mathrm{T}}$ and $(m, v, q, r)^{\mathrm{T}}$ of the minimization problems for the functionals (27) and (28) the constitutive law (26) is valid, i.e. the solutions of the variational problems above are related to each other via this material law. This can be seen analogously as in the 3-D-case, compare Theorem 3.8 in Duvaut and Lions (1976). We will use this relation among other things in the proof of Lemma 4.1.

By combining the two energy principles we are able to prove the following Lemma in analogy with the theorem of Prager and Synge (1947) (see also Theorem 2 in Alessandrini et al., 1998 and Theorem 3.7 in Schwab, 1995). It will play a crucial role in the proof of Theorem 4.1.

Lemma 4.1.:
For any quadruple of geometrically admissible quantities $(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)^{\mathrm{T}} \in V_{(1,1,2)}$ and for any 8-tuple of statically admissible stress resultants $(\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r})^{\mathrm{T}} \in \Sigma_{(1,1,2)}$, which are not necessarily related via the material law (26), the identity

$$\begin{aligned}
&\tfrac{1}{2}\mathscr{B}_{(1,1,2)}(w_0-\tilde{w}_0, w_2-\tilde{w}_2, \theta_x-\tilde{\theta}_x, \theta_y-\tilde{\theta}_y\,;\, w_0-\tilde{w}_0, w_2-\tilde{w}_2, \theta_x-\tilde{\theta}_x, \theta_y-\tilde{\theta}_y)\\
&\qquad+\tfrac{1}{2}\mathscr{A}_{(1,1,2)}([m-\tilde{m}, v-\tilde{v}q-\tilde{q}, r-\tilde{r}]\,;\,[m-\tilde{m}, v-\tilde{v}, q-\tilde{q}, r-\tilde{r}])\\
&\quad=\Pi_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)+\Pi^C_{(1,1,2)}([\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}])
\end{aligned}\tag{29}$$

is valid.

Proof:

From the principle of minimum potential energy, i.e. from the equivalence of variational and weak formulation we get

$$\mathscr{B}_{(1,1,2)}(w_0, w_2, \theta_x, \theta_y\,;\,\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y) = \mathscr{F}_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)$$

for any $(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)^{\mathrm{T}} \in V_{(1,1,2)}$. Analogously, we obtain

$$\mathscr{A}_{(1,1,2)}([m, v, q, r]\,;\,[\tilde{m}-m, \tilde{v}-v, \tilde{q}-q, \tilde{r}-r]) = 0$$

for any $(\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r})^{\mathrm{T}} \in \Sigma_{(1,1,2)}$ from the complementary energy principle. These relations are valid for any given geometrically and statically admissible quantities. From this the assertion follows in the same manner as in the 3-D-case, see e.g. Theorem 3.7 in Schwab (1995), since the left-hand side of (29) is equal to

$$\begin{aligned}
&[\tfrac{1}{2}\mathscr{B}_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y\,;\,\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)-\mathscr{B}_{(1,1,2)}(w_0, w_2, \theta_x, \theta_y\,;\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)\\
&\qquad+\tfrac{1}{2}\mathscr{A}_{(1,1,2)}([\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}]\,;\,[\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}])]+[\tfrac{1}{2}\mathscr{B}_{(1,1,2)}(w_0, w_2, \theta_x, \theta_y\,;\,w_0, w_2, \theta_x, \theta_y)\\
&\qquad-\mathscr{A}_{(1,1,2)}([m, v, q, r]\,;\,[\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}])+\tfrac{1}{2}\mathscr{A}_{(1,1,2)}([m, v, q, r]\,;\,[m, v, q, r])]\\
&\quad=[\Pi_{(1,1,2)}(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)+\Pi^C_{(1,1,2)}([\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}])]+0.
\end{aligned}$$

As mentioned before we prove Theorem 4.1 in the case of a plate which is hard clamped along its lateral side. So it must be assumed that the mid-plane $\omega \subset \mathbb{R}^2$ is such that the solution of the Dirichlet problem for Kirchhoff's plate eqn (20) is in the space $H^2_0(\omega) \bigcap H^3(\omega)$, which requires e.g. that ω is a convex polygon. (We refer to Blum and Rannacher (1980) and Melzer and Rannacher (1980) for details.) Moreover, we need variational formulations for the Kirchhoff model, analogously to (27) and (28). Since we are dealing with asymptotic analysis, the same scaling with $1/h^3$ as in the formulation for the (1,1,2)-model is necessary. The variational formulation for the Kirchhoff model corresponding to the principle of minimum potential energy reads:

Find $w_K \in H^2_0(\omega)$ such that w_K minimizes the functional

$$\Pi_K(w) = \int_\omega \left\{\frac{\mu(\lambda+\mu)}{6(\lambda+2\mu)}(\Delta w)^2 + \frac{\mu}{6}\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\} - gw\,\mathrm{d}x\,\mathrm{d}y$$

within $H^2_0(\omega)$ for given $g \in L^2(\omega)$. We see that the solution w_K does not depend on h. The space $V_K = H^2_0(\omega)$ is the space of geometrically admissible Kirchhoff deflections in the case of the hard clamped plate. In order to give a stress-related variational formulation, we consider the Kirchhoff stress resultants. The static equilibrium equations are given by

$$\frac{\partial m_{Kxx}}{\partial x}+\frac{\partial m_{Kxy}}{\partial y}-q_{Kx}=0$$
$$\frac{\partial m_{Kxy}}{\partial x}+\frac{\partial m_{Kyy}}{\partial y}-q_{Ky}=0$$
$$\frac{\partial q_{Kx}}{\partial x}+\frac{\partial q_{Ky}}{\partial y}+g=0 \tag{30}$$

The stress resultants $(m_K, q_K)^{\mathrm{T}} = (m_{Kxx}, m_{Kxy}, m_{Kyy}, q_x, q_{Ky})^{\mathrm{T}} \in [L^2(\omega)]^5$ satisfying these equilibrium conditions in the domain ω are called statically admissible Kirchhoff stress resultants in the hard clamped case. We denote by $\Sigma_K \subset [L^2(\omega)]^5$ the space of all statically admissible Kirchhoff stress resultants. The variational formulation for the Kirchhoff plate corresponding to the complementary energy principle states:

Find the 5-tuple $(m_K, q_K)^{\mathrm{T}} \in \Sigma_K$ minimizing the functional

$$\Pi_K^C(m,q) = \int_\omega \left\{\frac{3}{\mu}(m_{xx}^2+2m_{xy}^2+m_{yy}^2)-\frac{3\lambda}{\mu(3\lambda+2\mu)}(m_{xx}+m_{yy})^2\right\}\mathrm{d}x\,\mathrm{d}y$$

within the space Σ_K. By inserting the well-known relations for $(m_K, q_K)^{\mathrm{T}}$ in terms of w_K and using $\Pi_K(w_K) = -\frac{1}{2}\int_\omega g w_K\,\mathrm{d}x\,\mathrm{d}y$ we are able to formulate

Lemma 4.2.:

Let us denote by $w_K \in H_0^2(\omega)$ the minimizer of $\Pi_K(w)$ within $H_0^2(\omega)$ and by $(m_K, q_K)^{\mathrm{T}} \in \Sigma_K$ the minimizer of $\Pi_k^C(m,q)$ within Σ_K. These uniquely determined quantities thus represent the Kirchhoff solution. Then the identity

$$\Pi_K(w_K)+\Pi_K^C(m_K,q_K)=0 \tag{31}$$

is valid.

In the next step let us define the (1,1,2)-energy norm

$$\|\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y;[\tilde{m},\tilde{v},\tilde{q},\tilde{r}]\|^2_{E(1,1,2)} := \mathscr{B}_{(1,1,2)}(\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y;\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y) + \mathscr{A}_{(1,1,2)}([\tilde{m},\tilde{v},\tilde{q},\tilde{r}];[\tilde{m},\tilde{v},\tilde{q},\tilde{r}]) \tag{32}$$

for $(\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y;[\tilde{m},\tilde{v},\tilde{q},\tilde{r}])^{\mathrm{T}} \in V_{(1,1,2)} \times \Sigma_{(1,1,2)}$. From the fact that $\mathscr{B}_{(1,1,2)}$ and $\mathscr{A}_{(1,1,2)}$ are the energetic scalar products on $V_{(1,1,2)}$ and $\Sigma_{(1,1,2)}$ it is clear that

$$\|\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y\|^2_{e(1,1,2)} := \mathscr{B}_{(1,1,2)}(\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y;\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y)$$

and

$$\|[\tilde{m},\tilde{v},\tilde{q},\tilde{r}]\|^2_{c(1,1,2)} := \mathscr{A}_{(1,1,2)}([\tilde{m},\tilde{v},\tilde{q},\tilde{r}];[\tilde{m},\tilde{v},\tilde{q},\tilde{r}])$$

are the norms on $V_{(1,1,2)}$ and $\Sigma_{(1,1,2)}$, respectively, and thus; $\|\cdot\|_{E(1,1,2)}$ is indeed a norm on $V_{(1,1,2)} \times \Sigma_{(1,1,2)}$. So we have with Lemma 4.1 the identity

$$\|w_0-\tilde{w}_0,w_2-\tilde{w}_2,\theta_x-\tilde{\theta}_x,\theta_y-\tilde{\theta}_y;[m-\tilde{m},v-\tilde{v},q-\tilde{q},r-\tilde{r}]\|^2_{E(1,1,2)} = \|\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y;[\tilde{m},\tilde{v},\tilde{q},\tilde{r}]\|^2_{E(1,1,2)}-2\mathscr{F}_{(1,1,2)}(\tilde{w}_0,\tilde{w}_2,\tilde{\theta}_x,\tilde{\theta}_y), \tag{33}$$

whenever $(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y; [\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}])^{\mathrm{T}} \in V_{(1,1,2)} \times \Sigma_{(1,1,2)}$. Note that the right-hand side is independent of the solution of the (1,1,2)-model and, thus, the deviation to the Kirchhoff solution can be estimated without its explicit knowledge. Our aim is to give such an estimate. For this purpose geometrically and statically admissible functions must be found, which correspond to those of the Kirchhoff theory.

(1) *Geometrically admissible quantities*

The Euler–Lagrange equations of the (1,1,2)-variational model define a singular perturbation problem of an elliptic system for w_0, w_2, θ_x, θ_y if $h \to 0$. Formal expansion of its solution provides the leading terms which are used to choose the following geometrically admissible test functions

$$\tilde{w}_0 = w_K, \quad \tilde{\theta}_x = -\frac{\partial w_K}{\partial x}, \quad \tilde{\theta}_y = -\frac{\partial w_K}{\partial y}, \quad \tilde{w}_2 = \frac{\lambda}{2(\lambda+2\mu)}\Delta w_K + \underbrace{b(x,y)}_{\text{boundary corrector}}. \tag{34}$$

Now we see that $w_K \in H^3(\omega)$ is needed in order to have $\tilde{w}_2 \in H_0^1(\omega)$. Note that we have to ensure that for any $\delta > 0$ a boundary corrector $b(x,y) \in H^1(\omega)$ can be determined, such that $\tilde{w}_2 = 0$ on $\partial\omega$ and the inequalities

$$\int_\omega b^2(x,y)\,\mathrm{d}x\,\mathrm{d}y \leqslant C\delta \|\Delta w_K\|^2_{H^1(\omega)}, \quad \int_\omega |\nabla \tilde{w}_2|^2\,\mathrm{d}x\,\mathrm{d}y \leqslant C\delta^{-1}\|\Delta w_K\|^2_{H^1(\omega)} \tag{35}$$

hold. $C > 0$ is a constant independent of δ and w_K. Such a $b \in H^1(\omega)$ can always be found (compare Babuška and Pitkäranta, 1990 and Remark 3.12 in Schwab, 1995). Thus, we have ensured that the quadruple $(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y)^{\mathrm{T}}$ is in $V_{(1,1,2)}$, i.e. geometrically admissible. We remark that the expressions for $\tilde{w}_0$, $\tilde{\theta}_x$, $\tilde{\theta}_y$ and $\tilde{w}_2$ including the boundary correction term are exactly those which are obtained from the asymptotic analysis via singular perturbation theory for the 3-D problem (compare Nazarov and Zorin, 1989; Dauge and Gruais, 1996, 1998).

(2) *Statically admissible stress resultants*

We take as a basis for the statically admissible (1,1,2)-functions the corresponding 3-D-functions from Morgenstern (1959). We obtain

$$\tilde{m}_{\alpha\beta} = m_{K\alpha\beta}, \quad \tilde{v} = \frac{h^2}{10}g, \quad \tilde{q}_\alpha = q_{K\alpha}, \quad \tilde{r}_\alpha = \frac{h^2}{20}q_{K\alpha}, \quad \alpha, \beta \in \{x, y\}. \tag{36}$$

These stress resultants satisfy (25) with g instead of p. Hence, we have $(\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r})^{\mathrm{T}} \in \Sigma_{(1,1,2)}$, i.e. they are statically admissible.

Evaluating the right-hand side of the identity (33) for the geometrically and statically admissible quantities (34) and (36) and using (31) we obtain

$$\frac{1}{6}(\lambda+2\mu)\int_\omega b^2\,\mathrm{d}x\,\mathrm{d}y + \frac{\mu h^2}{80}\int_\omega |\nabla \tilde{w}_2|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$+\int_\omega \frac{h^2}{10}g\left(-\frac{6\lambda}{\mu(3\lambda+2\mu)}(m_{Kxx}+m_{Kyy}) + \frac{12(\lambda+\mu)}{\mu(3\lambda+2\mu)}\frac{h^2}{10}g\right)\mathrm{d}x\,\mathrm{d}y$$

$$+\int_\omega \frac{6h^2}{5\mu}(q_{Kx}^2+q_{Ky}^2)\,\mathrm{d}x\,\mathrm{d}y-\int_\omega \frac{h^2}{2}g\tilde{w}_2\,\mathrm{d}x\,\mathrm{d}y.$$

Let $g\in L^2(\omega)$ be given arbitrarily. Since w_K satisfies the Kirchhoff plate equation $[\mu(\lambda+\mu)]/[3(\lambda+2\mu)]\Delta^2 w_K = g$, a standard elliptic regularity estimate implies (see e.g. Blum and Rannacher, 1980)

$$\|\Delta w_K\|_{H^1(\omega)} \leqslant C_1(\omega)\frac{3(\lambda+2\mu)}{\mu(\lambda+\mu)}\|g\|_{L^2(\omega)},$$

$$\|\nabla(\Delta w_K)\|_{L^2(\omega)} \leqslant C_1(\omega)\frac{3(\lambda+2\mu)}{\mu(\lambda+\mu)}\|g\|_{L^2(\omega)}. \tag{37}$$

We collect the terms in the right-hand side of (33). The inequalities (37), the Cauchy–Schwarz-inequality and the choice $\delta = h/4$ in (35) enable us to estimate these terms separately as follows:

(1)

$$\frac{1}{6}(\lambda+2\mu)\int_\omega b^2\,\mathrm{d}x\,\mathrm{d}y+\frac{\mu h^2}{80}\int_\omega |\nabla\tilde{w}_2|^2\,\mathrm{d}x\,\mathrm{d}y-\int_\omega g\frac{h^2}{2}b\,\mathrm{d}x\,\mathrm{d}y$$
$$\leqslant\left(\frac{5\lambda+16\mu}{120}C_2(\omega)h+\frac{h^{5/2}}{4}C_3(\omega)\right)\|g\|^2_{L^2(\omega)}$$

(2)

$$-\int_\omega g\frac{h^2}{4}\frac{\lambda}{\lambda+2\mu}\Delta w_K\,\mathrm{d}x\,\mathrm{d}y+\int_\omega \frac{h^2}{10}g\left(-\frac{6\lambda}{\mu(3\lambda+2\mu)}(m_{Kxx}+m_{Kyy}\right)\mathrm{d}x\,\mathrm{d}y$$
$$=-\frac{3h^2}{20}\frac{\lambda}{\lambda+2\mu}\int_\omega g\Delta w_K\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{9h^2}{20}\frac{\lambda}{\mu(\lambda+\mu)}C_1(\omega)\|g\|^2_{L^2(\omega)}$$

(3)

$$\int_\omega \frac{h^4}{100}\frac{12(\lambda+\mu)}{\mu(3\lambda+2\mu)}g^2\,\mathrm{d}x\,\mathrm{d}y=\frac{3h^4(\lambda+\mu)}{25\mu(3\lambda+2\mu)}\|g\|^2_{L^2(\omega)}$$

(4)

$$\int_\omega \frac{6h^2}{5\mu}(q_{Kx}^2+q_{Ky}^2)\,\mathrm{d}x\,\mathrm{d}y=\int_\omega \frac{6h^2}{5\mu}\frac{\mu^2(\lambda+\mu)^2}{9(\lambda+2\mu)^2}|\nabla(\Delta w_K)|^2\,\mathrm{d}x\,\mathrm{d}y\leqslant\frac{6h^2}{5\mu}C_1^2(\omega)\|g\|^2_{L^2(\omega)}$$

Collecting these estimates results in the following theorem.

Theorem 4.1.:

Let $\omega\subset\mathbb{R}^2$ be such that the solution w_K of the Kirchhoff plate equation with the boundary condition that the plate is clamped along its lateral side is in $H_0^2(\omega)\bigcap H^3(\omega)$. Let

$(w_0, w_2, \theta_x, \theta_y; [m, v, q, r])^T \in V_{(1,1,2)} \times \Sigma_{(1,1,2)}$ be the solution of the (1,1,2)-plate-bending model and $(\tilde{w}_0, \tilde{w}_2, \tilde{\theta}_x, \tilde{\theta}_y; [\tilde{m}, \tilde{v}, \tilde{q}, \tilde{r}])^T \in V_{(1,1,2)} \times \Sigma_{(1,1,2)}$ the geometrically and statically admissible functions defined in (34) and (36). Then the estimate

$$\|w_0 - \tilde{w}_0, w_2 - \tilde{w}_2, \theta_x - \tilde{\theta}_x, \theta_y - \tilde{\theta}_y; [m - \tilde{m}, v - \tilde{v}, q - \tilde{q}, r - \tilde{r}]\|^2_{E(1,1,2)} = \mathcal{O}(h) \tag{38}$$

is valid, i.e. the solution of the (1,1,2)-model converges for $h \to 0$ with order $\sqrt{h}$ in the (1,1,2)-energy norm towards the Kirchhoff solution.

In Theorem 4.1. we estimated the difference between the solutions of the (1,1,2)-model and the Kirchhoff model in the (1,1,2)-energy norm. However, our original aim is an asymptotic estimate between the solution of the (1,1,2)-model and the 3-D solution in the 3-D-energy norm. Therefore, we characterize now the relation between the (1,1,2)-energy norm and the 3-D-energy norm. In order to find this relation with the help of the relations between the 3-D- and the (1,1,2)-bilinear forms, we have to introduce a scaled stress–strain-relation in analogy to the (1,1,2)-formulation. Let

$$\hat{A} = \frac{1}{h^3} A$$

with A given in (5). Its inverse exists for $h > 0$, $\mu > 0$ and $(3\lambda + 2\mu) > 0$. This linear mapping represents a scaled stress–strain-relation. Hooke's law then reads

$$\sigma = \hat{A}\varepsilon(u) \quad \text{or} \quad \varepsilon(u) = \hat{A}^{-1}\sigma.$$

We define the space of 3-D-geometrically admissible displacements

$$V_{3\text{-D}} = \mathscr{H}(\Omega) = \{u \in [H^1(\Omega)]^3 \,|\, u = 0 \text{ on } \Gamma_0\},$$

the scaled 3-D-load functional

$$\mathscr{F}_{3\text{-D}}(v) = \int_\omega \frac{1}{2} g(x, y) \left\{ v_3\left(x, y, \frac{h}{2}\right) + v_3\left(x, y, -\frac{h}{2}\right) \right\} \mathrm{d}x \, \mathrm{d}y$$

and the space of 3-D-statically admissible stresses

$$\Sigma_{3\text{-D}} = \left\{ \sigma \in [L^2(\Omega)]^{3\times 3}_{\text{sym}} \,\middle|\, \int_\omega \sigma : \varepsilon[v] \, \mathrm{d}V = \mathscr{F}(v) \forall v \in V_{3\text{-D}} \right\}.$$

We introduce the bilinear forms

$$\mathscr{A}_{3\text{-D}}(\sigma, \tau) = \int_\Omega \sigma : \hat{A}^{-1} \tau \, \mathrm{d}V \quad \text{and} \quad \mathscr{B}_{3\text{-D}}(u, v) = \int_\Omega \varepsilon[u] : \hat{A}\varepsilon[v] \, \mathrm{d}V,$$

which represent the energetic scalar products on $\Sigma_{3\text{-D}}$ and on $V_{3\text{-D}}$, respectively, and with their help the energy norm

$$\|u; \sigma\|^2_{E(3\text{-D})} = \mathscr{B}_{3\text{-D}}(u, u) + \mathscr{A}_{3\text{-D}}(\sigma, \sigma)$$

on $V_{3\text{-D}} \times \Sigma_{3\text{-D}}$.

$$\|u\|_{e(3\text{-D})} = \sqrt{\mathscr{B}_{3\text{-D}}(u, u)} \quad \text{and} \quad \|\sigma\|_{c(3\text{-D})} = \sqrt{\mathscr{A}_{3\text{-D}}(\sigma, \sigma)}$$

are the energy norm on $V_{3\text{-D}}$ and the complementary energy norm on $\Sigma_{3\text{-D}}$, respectively. Obviously, $\|\cdot\|_{e(3\text{-D})}$ is equivalent to $\|\cdot\|_{[H^1(\Omega)]^3}$ on $V_{3\text{-D}}$ by the Korn inequality and $\|\cdot\|_{c(3\text{-D})}$ is equivalent to $\|\cdot\|_{[L^2(\Omega)]^{3\times3}_{\text{sym}}}$ on $[L^2(\Omega)]^{3\times3}_{\text{sym}}$. In what follows we will give the choice of 3-D-displacements in terms of (1,1,2)-quantities and of 3-D-stresses in terms of (1,1,2)-stress resultants such that $\mathscr{B}_{3\text{-D}}$ and $\mathscr{A}_{3\text{-D}}$ coincide with $\mathscr{B}_{(1,1,2)}$ and $\mathscr{A}_{(1,1,2)}$, respectively. In the case of $\mathscr{B}_{3\text{-D}}$ this choice is straightforward. If we set

$$u_1^{112} = z\theta_x, \quad u_2^{112} = z\theta_y, \quad u_3^{112} = w_0 + z^2 w_2$$

we have

$$\|u^{112}\|^2_{e(3\text{-D})} = \|w_0, w_2, \theta_x, \theta_y\|^2_{e(1,1,2)}.$$

In the case $\mathscr{A}_{3\text{-D}}$ the choice is not so obvious. But with

$$\sigma_{11}^{112} = \frac{12}{h^3} z m_{xx}, \quad \sigma_{12}^{112} = \frac{12}{h^3} z m_{xy}, \quad \sigma_{13}^{112} = \frac{3}{2h} q_x - \frac{120}{h^5} z^2 r_x$$

$$\sigma_{22}^{112} = \frac{12}{h^3} z m_{yy}, \quad \sigma_{33}^{112} = \frac{12}{h^3} z v, \qquad \sigma_{23}^{112} = \frac{3}{2h} q_y - \frac{120}{h^5} z^2 r_y$$

$$\|\sigma^{112}\|^2_{c(3\text{-D})} = \|[m, v, q, r]\|^2_{c(1,1,2)}$$

is obtained, and indeed we are able to express the (1,1,2)-energy norm in terms of the 3-D-energy norm of suitable chosen displacements and stresses. Here u^{112} is a 3-D-geometrically admissible displacement, but unfortunately σ^{112} is only in $[L^2(\Omega)]^{3\times3}_{\text{sym}}$ and not statically admissible, since it does not satisfy the static 3-D-equilibrium eqns (1) and (4).

In order to estimate the difference between the 3-D solution and the solution of the (1,1,2)-model in the 3-D-energy norm with the help of the estimate of the difference between the (1,1,2)- and the Kirchhoff-quantities we have to introduce admissible displacements and stresses generated by quantities from the Kirchhoff theory:

(*a*) *Displacements*

$$u_1^K = -z\frac{\partial w_K}{\partial x}, \quad u_2^K = -z\frac{\partial w_K}{\partial y}$$

$$u_3^K = w_K + z^2\left(\frac{\lambda}{2(\lambda+2\mu)}\Delta w_K + \underbrace{b(x,y)}_{\text{boundary corrector}}\right),$$

where the boundary corrector $b(x, y)$ has to be chosen in such a way that u^K is 3-D-geometrically admissible and satisfies similar inequalities as in (35).

(*b*) *Stresses*

We have to introduce two slightly different Kirchhoff stresses

$$\sigma_{11}^K = \frac{12}{h^3} z m_{Kxx}, \quad \sigma_{22}^K = \frac{12}{h^3} z m_{Kyy}, \quad \sigma_{33}^K = \left(\frac{3}{2h} z - \frac{2}{h^3} z^3\right) g$$

$$\sigma_{12}^K = \frac{12}{h^3} z m_{Kxy}, \quad \sigma_{13}^K = \left(\frac{3}{2h} - \frac{6}{h^3} z^2\right) q_{Kx}, \quad \sigma_{23}^K = \left(\frac{3}{2h} - \frac{6}{h^3} z^2\right) q_{Ky}$$

and

$$\sigma_{ij}^{K}* = \sigma_{ij}^K \quad \text{for } i,j = 1,2,3, \quad i = j \neq 3, \quad \sigma_{33}^K* = 6/(5h)zg.$$

Only for $i = j = 3$ we have $\sigma_{ij}^K - \sigma_{ij}^{K}* \neq 0$ and hence

$$\|\sigma^K - \sigma^{K}*\|_{c(3\text{-D})}^2 = \frac{\lambda + \mu}{700\mu(3\lambda + 2\mu)} h^4 \|g\|_{L^2(\omega)}^2.$$

By using the triangle inequalities

$$\|u - u^{112}\|_{e(3\text{-D})} \leqslant \|u - u^K\|_{e(3\text{-D})} + \|u^{112} - u^K\|_{e(3\text{-D})}$$

and

$$\|\sigma - \sigma^{112}\|_{c(3\text{-D})} \leqslant \|\sigma - \sigma^K\|_{c(3\text{-D})} + \|\sigma^K - \sigma^{K}*\|_{c(3\text{-D})} + \|\sigma^{112} - \sigma^{K}*\|_{c(3\text{-D})}$$

and Theorem 4.1 as well as the results from Morgenstern (1959), we obtain

Theorem 4.2.:

Let $\omega \subset \mathbb{R}^2$ be such that the solution w_K of the Kirchhoff plate equation with the boundary condition that the plate is clamped along its lateral side is in $H_0^2(\omega) \bigcap H^3(\omega)$. Then with the above defined expressions the following asymptotic orders of the estimates are valid:

$$\|u - u^{112}\|_{e(3\text{-D})} = \mathcal{O}(\sqrt{h}) \quad \text{and} \quad \|\sigma - \sigma^{112}\|_{c(3\text{-D})} = \mathcal{O}(\sqrt{h}).$$

5. Numerical example

The (1,1,2)-plate model, as well as the corresponding 3-D shell formulation, are already established in numerical analysis of a certain class of problems. It has already been mentioned that the main merit of the (1,1,2)-model is the applicability of unmodified 3-D material laws. However, this application is not discussed in detail in this paper and therefore the reader is referred to Büchter et al. (1994), Betsch et al. (1996), for corresponding numerical investigations.

The example in this section addresses two main features of the (1,1,2)-model. Firstly, its convergence to the Kirchhoff and the 3-D solution and secondly its behavior in the range of thick plates.

We consider a quadratic plate under uniform loading, see Fig. 2. The load is assumed as dead load, thus it acts at each material point of the plate, not only on its top and bottom faces. Along the edges of the plate all displacements are zero ('fully clamped'), which is exactly the case for which the proof is given in the present paper. It is interesting, to remark that with these boundary conditions the thickness change of the plate is fixed at the edges in the 3-D-case, so that the transverse normal strains are zero. This leads to transverse normal stresses σ_{33} which apparently are neglected by the Mindlin theory.

For the calculation with the Mindlin model, one quarter of the plate is discretized by 12×12 8-

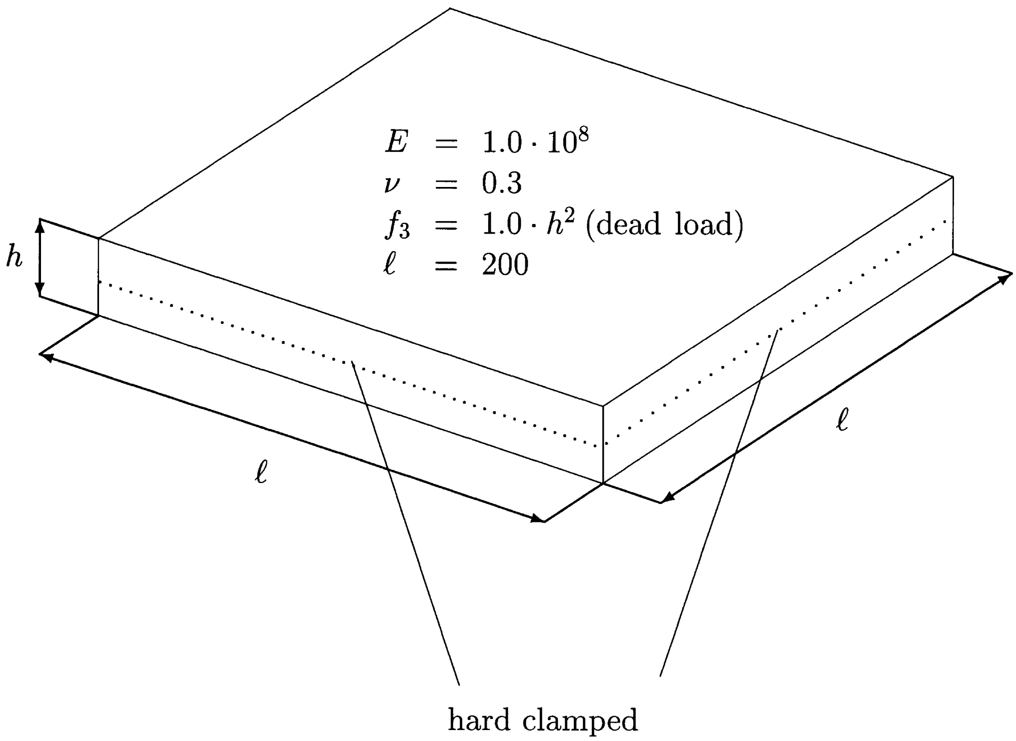

Fig. 2. Uniformly loaded square plate.

noded, reduced integrated plate elements, exploiting symmetry. For the 3-D-solution $10 \times 10 \times 4$ 20-noded, reduced integrated brick elements have been used. To obtain results for the (1,1,2)-model, the plate is discretized with one layer of 20-noded brick elements, thus modelling a quadratic distribution of the displacements across the thickness. It has been verified numerically that the densities of the meshes are such that the results are accurate enough for comparison. To investigate the asymptotic behavior of the described theories, the slenderness of the plate is varied from 1/2 up to 1/200, being aware of the fact that both values are outside the limits of practical significance and validity of the described plate theories.

In Fig. 3 the strain energy $\mathscr{B}(u, u)$, obtained by the numerical calculations, is plotted vs. the plate slenderness. Apparently all formulations tend to the same result as thickness decreases, which confirms the expositions in Section 4 (although a mathematical proof does not seem to need numerical confirmation anyway).

With increasing thickness, the Mindlin solution tends to overestimate the energy with respect to the 3-D-solution, i.e. it is too flexible. The decisive mechanical reason for this phenomenon is most likely the thickness constraint described before, which is not taken into account by the Mindlin model. Certainly, there is also a general deficit in grasping the 3-D overall behavior of the structure, because in the case of thick plates, there are not only 'bending' and 'shear' deformations, but a general 3-D stress and strain state has to be taken into consideration. However, the neglection of these 'higher order' influences on the structural behavior does not necessarily lead to a 'softening'

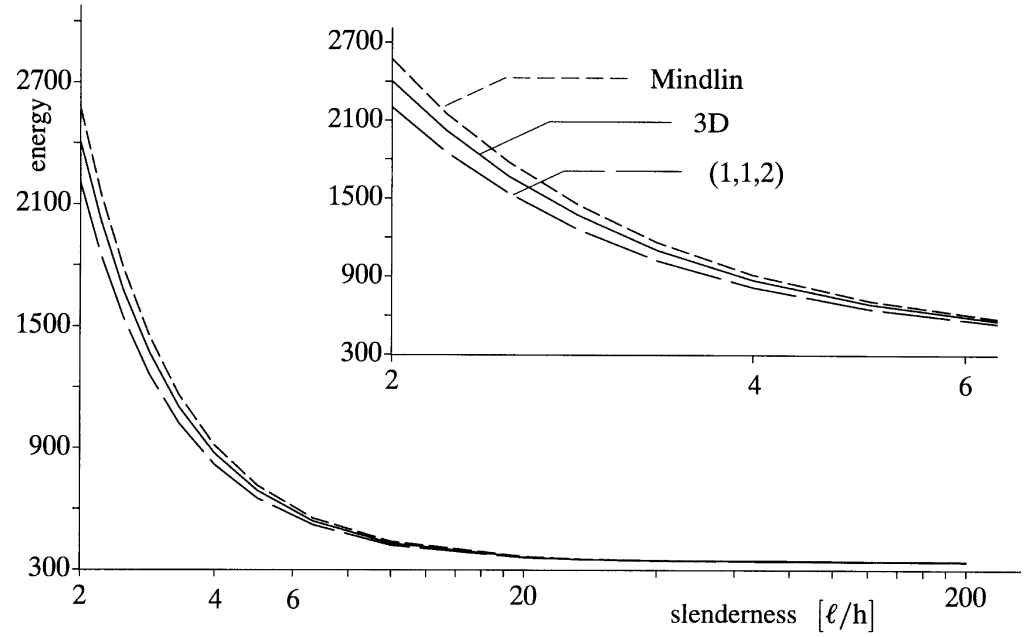

Fig. 3. Strain energy vs. plate slenderness.

effect and therefore an overestimation of the energy. In fact, for a similar problem with boundary conditions which do not lead to a thickness constraint at the edges, the Mindlin model underestimates the 3-D solution, i.e. it behaves too stiff.

The (1,1,2)-model again behaves 'stiffer', which is easy to explain. Firstly, it is able to describe at least approximately the 3-D behavior, especially the thickness constraint at the edges. Secondly, it can be interpreted as a 3-D finite element solution, obtained with a coarse mesh in thickness direction. As the energy in displacement models approaches the exact value from below, any other result would be a surprise.

6. Conclusions

The (1,1,2)-plate model has been put into the framework of hierarchical modelling using energy projection. It has been shown for a certain boundary condition (hard clamped) that the model is asymptotically correct, without a need to modify the constitutive law (a similar proof can be given for alternative boundary conditions, but is not reproduced herein).

The investigation points to two main conclusions. Firstly, the mathematical foundation of a plate model including thickness stretch, frequently used in finite element plate (and shell) analysis is given. Thus, a mathematically sound plate model is available for efficient numerical analyses, using fully 3-D constitutive laws. Secondly, it could be seen that the (1,1,2)-model is in fact the lowest one in a family of models that can be used together with unmodified constitutive laws.

Thus, the optimal compromise between 2-D-efficiency and 3-D-accuracy might be found for a certain class of problems.

The extension of the present study to shell analysis and geometrical non-linearity is straightforward. Some additional considerations have to be made in the formulation of the hierarchical model, because membrane and bending action are no longer decoupled in these cases. In addition, the evaluation of the boundary correction terms might cause some trouble. Nevertheless, Morgenstern's idea of using the Prager–Synge theorem for the proof still works. For results on this subject see e.g. Koiter (1970) and Destuynder (1997).

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